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In-medium propagation of particles in an Open Quantum System approach

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Abstract. The quark–gluon plasma (QGP), a deconfined state of quarks and gluons created at extreme temperatures and densities, provides unique insight into the properties of the strong force and potentially the early universe. However, current formalisms to describe the QGP are artificial, limited and their complexity scales considerably when attempting to lift approximations. Recently, the idea to apply the formalism of Open Quantum Systems (OQS) to this problem has emerged as a more natural approach, since partons propagating in the QGP and experiencing its influence can be modeled as quantum systems interacting with an environment. In this work, after an overview of the theory of OQS, we use the Markovian Lindblad Equation to describe the propagation of partons in the QGP, namely their splitting dynamics. First, we develop a phenomenological model of splitting using the Lindblad equation. Then, these results are generalized to the case where partons have spin, color and momentum. Finally, we attempt to derive this splitting model from a microscopic interaction Hamiltonian describing parton-QGP interaction. These results allow us to obtain expressions for the splitting times and potentially assess the modifications induced by the QGP relative to vacuum propagation.

Keywords: QCD, Quark-Gluon Plasma, Open Quantum Systems

1 Introduction

The Quark–Gluon Plasma (QGP) is a state of matter characterized by extremely high energies in which quarks and gluons become deconfined [1]. Its existence was first theorized in the late 1970s as a phase that occurred for a few milliseconds after the Big Bang. In 2000, experiments at CERN provided the first evidence for this state of matter, and since then the QGP has remained a central subjects of study in high-energy nuclear physics.

The current state-of-the-art formalism describing the propagation of jet particles through the QGP is known as the BDMPS-Z framework [2]. Although highly successful, it relies on significant simplifying assumptions: full analytical solutions require an eikonal assumption which the coordinates at play are assmued to be straight lines. Relaxing these constraints drastically increases the mathematical complexity of the model, making numerical solutions very challenging.

This motivates alternative approaches, such as the Open Quantum Systems (OQS) framework. In this perspective, the jet can be treated as a quantum system interacting with the QGP, which plays the role of a thermal environment. Although still at a very early phase, it may offer a theory more natural and flexible of jet–medium interactions.

In this work, we present the key mathematical elements and physical concepts underlying the general OQS formalism, and then introduce simplified toy models for parton splitting processes inspired by the physics of the QGP.

2 Theoretical foundations

2.1 The Density Operator Formalism

In the standard formalism of quantum mechanics, the state of a system is represented by a unitary vector in a separable Hilbert space. However, this description fails whenever the system cannot be assigned a single state vector, such as in the case of statistical mixtures of different quantum states. In this case, there is a statistical uncertainty in the state of the system and the use of a statistical ensemble is necessary. This could occur, for instance, due to an imprecise preparation of a quantum state or be a result of its interaction with an environment too complex to describe exactly. These states are called mixed states, whereas the cases where the system's state is known with certainty are referred to as pure states. The need to properly describe such mixed states becomes especially important in the study of open quantum systems, where interactions with an environment prevent a pure state description. We therefore introduce the density operator formalism, which can describe not only pure but also mixed states within a single framework.

In this formalism, the state of the system is described by a self-adjoint linear operator $\rho : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space over $\mathbb{C}$, satisfying $\mathrm{tr}(\rho) = 1$ and $\langle\psi|\rho|\psi\rangle \geq 0 \quad \forall\, \psi \in \mathcal{H}$. The unit vectors that previously described the state of the system are now identified with density operators of the form $\rho = |\psi\rangle\langle\psi|$, in which case the system is in a pure state. Conversely, if we have a set of pure states $|\psi_i\rangle\langle\psi_i|$, each with a probability p_i of occuring in the statistical ensemble, then this ensemble is described by the density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, in which case the system is in a mixed state. Given an observable $\hat{A} : \mathcal{H} \to \mathcal{H}$, the expectation value in the state ρ is $\langle\hat{A}\rangle = \mathrm{tr}(\hat{A}\rho)$. Finally, the time evolution of the density operator is governed by the Liouville-von Neumann equation:

$$i\hbar\frac{d\rho}{dt} = [\hat{H}, \rho], \tag{1}$$

where $\hat{H}$ is the system Hamiltonian. Consequently, given the density operator at an initial time t_0, the state of the system at t_1 is given by:

$$\rho(t_1) = U(t_1, t_0)\rho(t_0)U^{\dagger}(t_1, t_0) \tag{2}$$

where $U(t_1, t_0) = \mathcal{T} e^{-\frac{i}{\hbar}\int_{t_0}^{t_1} H(t)dt}$ is the time evolution operator [3]. For a two-level system, the following result holds:

Lemma 2.1. If $\rho \in M_{2\times2}(\mathbb{C})$ satisfies $\mathrm{tr}(\rho) = 1$, is self-adjoint and $\langle\phi|\rho|\phi\rangle \geq 0 \quad \forall\, \phi \in \mathbb{C}^2$, then ρ is of the form:

$$\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}, \quad a \in [0,1], \quad b \in \mathbb{C} \text{ and } |b|^2 \leq a(1-a)$$

Proof. Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{C})$. $\mathrm{tr}(\rho) = 1 \implies d = 1-a$. ρ is self-adjoint $\implies a, d \in \mathbb{R}$ and $c = b^*$. $\langle\phi|\rho|\phi\rangle \geq 0 \quad \forall\, \phi \in \mathbb{C}^2 \implies a \geq 0$ (taking $\phi = (1,0)$), $d = 1-a \geq 0$ (taking $\phi = (0,1)$) and $\langle\phi|\rho|\phi\rangle \geq 0 \ \forall\phi \in \mathbb{C}^2 \implies a \in [0,1]$ and $ad - bc = a(1-a) - |b|^2 \geq 0$ ($\langle\phi|\rho|\phi\rangle \geq 0 \ \forall\phi \in \mathbb{C}^2 \implies \det\rho \geq 0$). □

2.2 Time Evolution in Open Quantum Systems

To describe an open quantum system, we consider that it is part of a larger isolated system, which, besides the open system whose dynamics we wish to study, is also composed of a much larger bath. Hence, the Hilbert space of the global closed system is the tensor product of two smaller Hilbert spaces, one corresponding to the open system, and the other to the bath: $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. While in theory we could simply use the von Neumann equation to solve the dynamics of the density matrix in the global Hilbert space, in practice, the bath is extremely complex, rendering this equation unsolvable. The goal of the theory of open quantum systems is to describe the time evolution of ρ_S, which is related to the global density matrix through the partial trace over the bath $\rho_S = \mathrm{tr}_B(\rho)$. Thus, given ρ_S at an instant t_0, the state of the system at time t_1 is given by

$$\rho_S(t_1) = \mathrm{tr}_B(U(t_1,t_0)\rho(t_0)U^\dagger(t_1,t_0)), \tag{3}$$

where we used the fact that $\mathcal{H}_S \otimes \mathcal{H}_B$ is a closed system, so its evolution is given by equation 2. We want to rewrite this expression in the form $\rho_S(t_1) = \mathcal{E}_{(t_1,t_0)}[\rho_S(t_0)]$, where we call the superoperator $\mathcal{E}_{(t_1,t_0)}$, which connects ρ_S at different times, a dynamical map. It can be shown [4] that we can always write:

$$\rho_S(t_1) = \sum_\alpha K_\alpha(t_1,t_0,\rho_S)\,\rho_S(t_0)\,K_\alpha^\dagger(t_1,t_0,\rho_S) \tag{4}$$

where K_α are operators which depend on the state ρ_S at time t_0.

On the other hand, if we write the initial state as $\rho(t_0) = \rho_S(t_0)\otimes\rho_B(t_0) + \delta\rho(t_0)$, where the second term contains the correlations between the system and the bath at the initial time, we find an alternative form for the dynamical map:

$$\begin{aligned}\rho_S(t_1) &= \mathrm{Tr}_B\{U(t_1,t_0)[\rho_S(t_0)\otimes\rho_B(t_0) + \rho_{\mathrm{corr}}(t_0)]U^\dagger(t_1,t_0)\} \\ &= \sum_i \lambda_i \,\mathrm{Tr}_B\{U(t_1,t_0)[\rho_S(t_0)\otimes|\psi_i\rangle\langle\psi_i|]U^\dagger(t_1,t_0)\} \\ &\quad + \mathrm{Tr}_B[U(t_1,t_0)\rho_{\mathrm{corr}}(t_0)U^\dagger(t_1,t_0)] \\ &= \sum_\alpha K_\alpha(t_1,t_0)\rho_S(t_0)K_\alpha^\dagger(t_1,t_0) + \delta\rho(t_1,t_0) \\ &\equiv \mathcal{E}_{(t_1,t_0)}[\rho_S(t_0)],\end{aligned} \tag{5}$$

where $\alpha = \{i,j\}$, $K_{i,j}(t_1,t_0) = \sqrt{\lambda_i}\,\langle\psi_j|U(t_1,t_0)|\psi_i\rangle$ and we have used the spectral decomposition of $\rho_B(t_0) = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$.

A universal dynamical map (UDM) is a dynamical map which is independent of the state it acts upon:

$$\rho_S(t_1) = \sum_\alpha K_\alpha(t_1,t_0)\rho_S(t_0)K_\alpha^\dagger(t_1,t_0). \tag{6}$$

In this case K_α are called the Kraus operators.

It is thus easy to see, and it can be proven rigorously [4], that a dynamical map being a UDM is equivalent to having an initial condition where $\delta\rho(t_0) = 0$, that is the system and bath are initially uncorrelated.

An interesting property is that the inverse of a UDM is only a UDM if it is unitary, which only occurs if the system and bath are decoupled. This means the time evolution of open quantum systems is, in general, irreversible.

2.3 Markovian Evolution: the Lindblad Equation

We could naively assume that, given the UDM that describes the evolution of a system between instants t_0 and t_2, $\mathcal{E}_{(t_2,t_0)}$, then it is possible, for every intermediate instant t_1, to find two UDMs, $\mathcal{E}_{(t_2,t_1)}$ and $\mathcal{E}_{(t_1,t_0)}$, such that: $\mathcal{E}_{(t_2,t_0)} = \mathcal{E}_{(t_2,t_1)}\mathcal{E}_{(t_1,t_0)}$.

However, even if in the initial state the system and bath are uncorrelated, $\rho(t_0) = \rho_S(t_0)\otimes\rho_B(t_0)$, the state at time t_1 will, in general, exhibit correlations, $\delta\rho(t_1) \neq 0$, due to the system-bath interaction that occurs in this time interval. Consequently, even if $\mathcal{E}_{(t_1,t_0)}$ and $\mathcal{E}_{(t_2,t_0)}$ are UDMs, $\mathcal{E}_{(t_2,t_1)}$ is not. Therefore, UDMs in general do not satisfy this simple continuity condition. The ones that do have a special designation:

Definition 2.2. A quantum system undergoes a Markovian evolution if the composition law for UDMs applies: $\mathcal{E}_{(t_2,t_0)} = \mathcal{E}_{(t_2,t_1)}\mathcal{E}_{(t_1,t_0)}$. This property is referred to as the divisibility condition.

In general, the evolution of an open quantum system is not Markovian as typically correlations between the system and bath develop over time. However if the correlation term $\delta\rho$ does not affect the dynamics significantly, then a Markovian evolution serves as a good approximation.

As mentioned above, we are typically interested in obtaining a differential equation for ρ_S that governs the dynamics of the open quantum system. This equation is called a Master Equation and it is related to the form of the UDM:

$$\begin{aligned}\frac{d\rho(t)}{dt} &= \lim_{\epsilon\to 0}\frac{\rho(t+\epsilon)-\rho(t)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{[\mathcal{E}_{(t+\epsilon,t)}-\mathbb{1}]}{\epsilon}\rho(t) = \mathcal{L}_t\rho(t)\end{aligned} \tag{7}$$

When the time evolution is Markovian the master equation has necessarily a very specific form called the Lindblad equation as follows in Theorem 2.3.

Theorem 2.3. A differential equation is a Markovian master equation if and only if it can be written in the form

$$\begin{aligned}\frac{d\rho(t)}{dt} &= -\frac{i}{\hbar}[H(t),\rho(t)]\\ &+ \sum_k \gamma_k(t)\left[V_k(t)\rho(t)V_k^\dagger(t) - \frac{1}{2}\{V_k^\dagger(t)V_k(t),\rho(t)\}\right]\end{aligned} \tag{8}$$

where $H(t)$ and $V_k(t)$ are time-dependent operators (the latter called jump operators), with $H(t)$ self-adjoint, and $\gamma_k(t) \geq 0$ for every k and time t.

It is possible to arrive at the Lindblad equation through more physical arguments and approximations that are valid in a certain regime. This also tells us in which physical conditions it is justified to approximate the evolution as Markovian. One such situation is the weak coupling regime, where the interaction between the system and the bath is considered to be small. We write the Hamiltonian as a sum of the system and bath Hamiltonians, plus an interaction between them: $H = H_S + H_B + V$. Then, we assume we can write $V = \sum_k A_k \otimes B_k$, with $A_k^\dagger = A_k$ and $B_k^\dagger = B_k$.
Under these assumptions, it is possible to arrive at a Markovian equation for the system evolution:

$$\begin{aligned}\frac{d\rho_S(t)}{dt} &= -\frac{i}{\hbar}[H_S + H_{LS},\rho_S(t)]\\ &+ \sum_\omega \sum_{k,\ell}\gamma_{k\ell}(\omega)\Big[A_\ell(\omega)\rho_S(t)A_k^\dagger(\omega)\\ &\qquad - \tfrac{1}{2}\{A_k^\dagger(\omega)A_\ell(\omega),\rho_S(t)\}\Big],\end{aligned}$$

where the jump operators are given by

$$A_k(\omega) = \sum_{\epsilon'-\epsilon=\omega} |\psi_\epsilon\rangle\langle\psi_\epsilon| A_k |\psi_{\epsilon'}\rangle\langle\psi_{\epsilon'}|, \tag{9}$$

where $|\psi_\epsilon\rangle$ are the eigenstates of H_S with energy ϵ. The other quantities are given by the following expressions:

$$\gamma_{kl} = 2\pi \mathrm{tr}(B_k(\omega)B_l\rho_{th}), \tag{10}$$

$$H_{LS} = \sum_\omega \sum_{k,\ell} S_{k\ell}(\omega)A_k^\dagger(\omega)A_\ell(\omega), \tag{11}$$

$$S_{kl}(\omega) = \text{P.V.}\int_{-a}^{a}\frac{\mathrm{Tr}\left[B_k(\omega')B_l\rho_B\right]}{(\omega-\omega')}\,d\omega', \tag{12}$$

where $B_k(\omega)$ is the Fourier transform of B_k in the interaction picture.
In the remaining of this work, we will work under this Markovian approximation and, as such, we shall use the Lindblad equation to determine the dynamics of the open quantum systems under study.

3 Phenomenological Models of Splitting

3.1 Basic Dynamics of Splitting Phenomena

A simple prototype of a $1 \to 2$ particle splitting phenomenon is a system S described by the 2-dimensional Hilbert space spanned by $\{|1\rangle, |2\rangle\}$, where $|1\rangle$ corresponds to a 1-particle state and $|2\rangle$ to a 2-particle state, with the usual inner product $\langle i|j\rangle = \delta_{ij}$. We will impose energy conservation in the splitting, that is we consider that both states are eigenvectors of the system Hamiltonian, H_S, sharing the same eigenvalue ϵ. The simple problem under study will only allow the transition from state $|1\rangle$ to state $|2\rangle$, via interaction with a medium, the nature of which will be encoded in the constant γ. Consequently, the process will be ruled by one jump operator, L, and an Hamiltonian H_S, respectively given by, in the $\{|1\rangle, |2\rangle\}$ basis,

$$L = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \qquad H_S = \varepsilon\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{13}$$

Denoting ρ as the density matrix of the system S, the Lindblad equation reduces to

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H_S,\rho] + \gamma\left(L\rho L^\dagger - \frac{1}{2}\{L^\dagger L,\rho\}\right). \tag{14}$$

Since H_S is proportional to the identity, it will always commute with ρ, hence $[H_s,\rho] = 0$. Lemma 2.1 gives us the general form of the system's density matrix with which, together with the expression for L, we find

$$L\rho L^\dagger = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}a(t) & b(t)\\ b(t)^* & 1-a(t)\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & a(t)\end{pmatrix} \tag{15}$$

$$\{L^\dagger L,\rho\} = \begin{pmatrix}2a(t) & b(t)\\ b(t) & 0\end{pmatrix}. \tag{16}$$

Using these determined expressions, the Lindblad equation reduces to

$$\frac{\partial\rho}{\partial t} \equiv \begin{pmatrix}\frac{\partial a}{\partial t}(t) & \frac{\partial b}{\partial t}(t)\\ \frac{\partial b^*}{\partial t}(t) & -\frac{\partial a}{\partial t}(t)\end{pmatrix} = \gamma\begin{pmatrix}-a(t) & -\frac{b(t)}{2}\\ -\frac{b(t)^*}{2} & a(t)\end{pmatrix}, \tag{17}$$

from which we get two (uncoupled) equations: $\dot{a}(t) = -\gamma\, a(t)$ and $\dot{b}(t) = -\frac{\gamma}{2}\, b(t)$. Imposing the initial condition $\rho(0) = |1\rangle\langle 1|$ (which implies in particular that $b(t) = 0$), we obtain the solution

$$\rho(t) = \begin{pmatrix}e^{-\gamma t} & 0\\ 0 & 1-e^{-\gamma t}\end{pmatrix}. \tag{18}$$

In this solution, $a(t)$ is the probability of finding the system in the state $|1\rangle\langle 1|$. One can also compute the entropy of the system, which is found to be given by

$$\begin{aligned}S(t) &= -\mathrm{tr}(\rho(t)\log\rho(t))\\ &= -(-\gamma t\, e^{-\gamma t} + (1-e^{-\gamma t})\log(1-e^{-\gamma t}))\\ &= \gamma t\, e^{-\gamma t} - (1-e^{-\gamma t})\log(1-e^{-\gamma t}).\end{aligned} \tag{19}$$

Its plot is depicted in Figure 1.

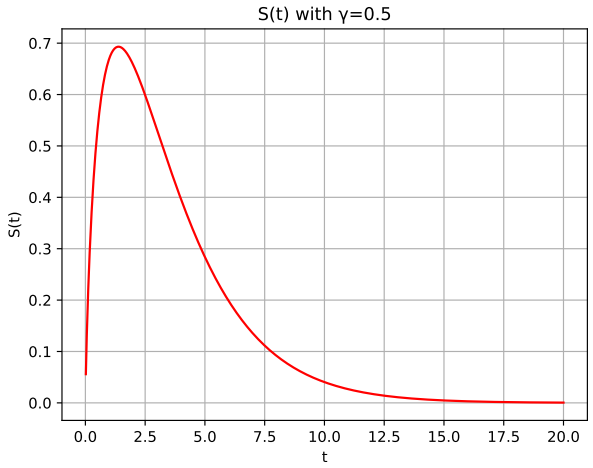

Figure 1. Entropy $S(t)$, taking $\gamma = 0.5$, $t \in [0, 20]$

3.2 Spin and Colour Dynamics in Parton Splitting

A possible extension to the splitting model described above consists in considering that the particles involved have spin, in which case it is necessary to impose spin conservation. We consider that the initial parton is a quark (q), a spin$-1/2$ fermion, and that it splits into another quark and a gluon (g), a spin-1 boson. This process is depicted in figure 2.

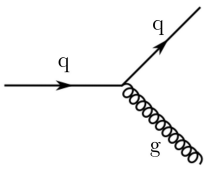

Figure 2. Feynman Diagram of the splitting $q \to qg$

If the initial quark is in the state $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$, the final particles will need to be in the same total spin state. The two possible total spin states can be written in terms of the spin state of the individual particles:

$$\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle = \pm\sqrt{\frac{2}{3}}\left|\frac{1}{2}, \mp\frac{1}{2}\right\rangle \otimes |1, \pm 1\rangle \mp \sqrt{\frac{1}{3}}\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \otimes |1, 0\rangle \tag{20}$$

Thus, the new Hilbert space will be the span of $\{|1,\uparrow\rangle, |1,\downarrow\rangle, |2,\uparrow\rangle, |2,\downarrow\rangle\}$, where $|1\rangle$ and $|2\rangle$ indicate the number of particles and $|\uparrow\rangle = \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$ e $|\downarrow\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ the total spin state. Considering that the two spin states have the same energy and imposing again that the energy is conserved during the splitting, the Hamiltonian is once again a multiple of the identity: $H_S = \varepsilon\mathbb{1}$. When splitting occurs, there will only be transitions $|1,\uparrow\rangle \to |2,\uparrow\rangle$ and $|1,\downarrow\rangle \to |2,\downarrow\rangle$. Consequently, we consider 2 jump operators, whose representation in the $\{|1,\uparrow\rangle, |1,\downarrow\rangle, |2,\uparrow\rangle, |2,\downarrow\rangle\}$ basis is

$$\begin{aligned} L_\uparrow &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbb{L}_\uparrow & \mathbf{0} \end{pmatrix}, \quad \mathbb{L}_\uparrow = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ L_\downarrow &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbb{L}_\downarrow & \mathbf{0} \end{pmatrix}, \quad \mathbb{L}_\downarrow = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{21}$$

These are associated to constants $\gamma_\uparrow$ and $\gamma_\downarrow$ respectively. The density matrix can be written by blocks:

$$\begin{aligned} \rho(t) = \begin{pmatrix} A(t) & B(t) \\ B(t)^\dagger & C(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ b(t)^* & c(t) \end{pmatrix}, \\ C(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t)^* & 1 - a(t) - c(t) - \alpha(t) \end{pmatrix} \end{aligned} \tag{22}$$

The initial conditions are $C(0) = B(0) = 0$ (since, if initially there is only one particle, then the system can only be in the span of $|1,\uparrow\rangle$ and $|1,\downarrow\rangle$) and $A(0) = \rho_i$ where ρ_i is a 2×2 density matrix which contains information on the quantum superpositions or statistical mixtures that between $|1,\uparrow\rangle$ e $|1,\downarrow\rangle$ in the initial state. We can express this matrix, as before, in the form:

$$\rho_i = \begin{pmatrix} a(0) & b(0) \\ b(0)^* & 1 - a(0) \end{pmatrix} \tag{23}$$

The Lindblad equation is:

$$\begin{aligned} \frac{\partial\rho}{\partial t} = &-\frac{i}{\hbar}[H_S, \rho] + \gamma_\uparrow\left(L_\uparrow\rho L_\uparrow^\dagger - \frac{1}{2}\{L_\uparrow^\dagger L_\uparrow, \rho\}\right) \\ &+ \gamma_\downarrow\left(L_\downarrow\rho L_\downarrow^\dagger - \frac{1}{2}\{L_\downarrow^\dagger L_\downarrow, \rho\}\right) \end{aligned} \tag{24}$$

Since H_S is proportional do the identity, the commutator with ρ is 0. It remains to determine the terms:

$$\begin{aligned} L_\uparrow\rho L_\uparrow^\dagger &= \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbb{L}_\uparrow & \mathbf{0} \end{pmatrix}\begin{pmatrix} A(t) & B(t) \\ B(t)^\dagger & C(t) \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbb{L}_\uparrow \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{L}_\uparrow A(t)\mathbb{L}_\uparrow \end{pmatrix} \\ \{L_\uparrow^\dagger L_\uparrow, \rho\} &= \begin{pmatrix} \{\mathbb{L}_\uparrow, A(t)\} & \mathbb{L}_\uparrow B(t) \\ B(t)^\dagger\mathbb{L}_\uparrow & \mathbf{0} \end{pmatrix} \end{aligned} \tag{25}$$

The results are analogous for $L_\downarrow$. If we define:

$$\gamma = \begin{pmatrix} \gamma_\uparrow & 0 \\ 0 & \gamma_\downarrow \end{pmatrix} \tag{26}$$

then the Lindblad equation is reduced to:

$$\frac{\partial\rho}{\partial t} = \begin{pmatrix} \frac{\partial A}{\partial t}(t) & \frac{\partial B}{\partial t}(t) \\ \frac{\partial B^\dagger}{\partial t}(t) & \frac{\partial C}{\partial t}(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\{\gamma, A(t)\} & -\frac{1}{2}\gamma B(t) \\ -\frac{1}{2}B(t)^\dagger\gamma & \gamma\,\mathrm{diag}(a(t), c(t)) \end{pmatrix} \tag{27}$$

We first analyze the equation for $A(t)$:

$$\frac{\partial A}{\partial t} = \begin{pmatrix} \frac{\partial a}{\partial t}(t) & \frac{\partial b}{\partial t}(t) \\ \frac{\partial b^*}{\partial t}(t) & \frac{\partial c}{\partial t}(t) \end{pmatrix} = \begin{pmatrix} -\gamma_\uparrow a(t) & -\frac{\gamma_\uparrow+\gamma_\downarrow}{2}b(t) \\ -\frac{\gamma_\uparrow+\gamma_\downarrow}{2}b(t)^* & -\gamma_\downarrow c(t) \end{pmatrix} \tag{28}$$

$$\implies A(t) = \begin{pmatrix} e^{-\gamma_\uparrow t}a(0) & e^{-\frac{\gamma_\uparrow+\gamma_\downarrow}{2}}b(0) \\ e^{-\frac{\gamma_\uparrow+\gamma_\downarrow}{2}t}b(0)^* & e^{-\gamma_\downarrow t}(1 - a(0)) \end{pmatrix} \tag{29}$$

Thus, both the populations and the coherences of the single-particle states decay exponentially, with the decay rates evident above. Secondly, the equation for $B(t)$ would also result in an exponential decay (with decay rate $\frac{\gamma_\uparrow}{2}$ or $\frac{\gamma_\downarrow}{2}$ depending on the matrix element). Since the initial condition is $B(0) = 0$, it follows that $B(t) = 0 \quad \forall t$. Lastly, the equation for $C(t)$ is:

$$\frac{\partial C}{\partial t} = \begin{pmatrix} \frac{\partial \alpha}{\partial t}(t) & \frac{\partial \beta}{\partial t}(t) \\ \frac{\partial \beta^*}{\partial t}(t) & -\frac{\partial a}{\partial t}(t) - \frac{\partial c}{\partial t}(t) - \frac{\partial \alpha}{\partial t}(t) \end{pmatrix} = \begin{pmatrix} \gamma_\uparrow a(t) & 0 \\ 0 & \gamma_\downarrow c(t) \end{pmatrix} \tag{30}$$

$$\implies C(t) = \begin{pmatrix} (1 - e^{-\gamma_\uparrow t})a(0) & 0 \\ 0 & (1 - e^{-\gamma_\downarrow t})(1 - a(0)) \end{pmatrix} \tag{31}$$

Hence, when $t \to \infty$, the system tends to the states with 2 particles, with the statistical distribution between these two states matching the initial spin distribution. Note that $\beta(t) = 0$, which means that the system always tends to a non-coherent statistical mixture of $|2, \uparrow\rangle$ e $|2, \downarrow\rangle$ - even if there are quantum superpositions in the initial state, they will never be present in the final state. For instance, if the initial state is a pure eigenstate of σ_x, such as $|+\rangle = \frac{1}{\sqrt{2}}(|1, \uparrow\rangle + |1, \downarrow\rangle)$, the final state will not be the pure state $\frac{1}{\sqrt{2}}(|2, \uparrow\rangle + |2, \downarrow\rangle)$, but a statistical mixture, $\rho_f = \frac{1}{2}(|2, \uparrow\rangle\langle 2, \uparrow| + |2, \downarrow\rangle\langle 2, \downarrow|)$. This arises from the fact that we are considering two jump operators identified by the projection of the spin along z. As a result, when there is a transition, the action of these operators effectively functions as a measurement of the spin's z-component, collapsing the system into the $|\uparrow\rangle$ or $|\downarrow\rangle$ states (each with probability 1/2), thereby producing the corresponding statistical mixture. The final expression for the density matrix is:

$$\rho(t) = \left(\begin{array}{cc|cc} e^{-\gamma_\uparrow t}a(0) & e^{-\frac{\gamma_\uparrow+\gamma_\downarrow}{2}}b(0) & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ e^{-\frac{\gamma_\uparrow+\gamma_\downarrow}{2}t}b(0)^* & e^{-\gamma_\downarrow t}(1 - a(0)) & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & (1 - e^{-\gamma_\uparrow t})a(0) & 0 \\ & & 0 & (1 - e^{-\gamma_\downarrow t})(1 - a(0)) \end{array}\right) \tag{32}$$

This model can be easily extended to the case where we consider partons with color (red (r), green (g) or blue (b)). In this case, we need to impose conservation of color in the splitting process. Since there are three possible colors, the Hilbert space will now be 6-dimensional. It is spanned by $\{|1, r\rangle, |1, g\rangle, |1, b\rangle, |2, r\rangle, |2, g\rangle, |2, b\rangle\}$, where $|1\rangle$ and $|2\rangle$ indicate the number of particles and $|i\rangle$ identifies the total color state. As before, $|2, i\rangle$ corresponds to a linear combination of individual color states of the 2 particles (quark and gluon), such that that the total color is i. With $i = r, g$ or b, we define three jump operators:

$$L_i = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbb{L}_i & \mathbf{0} \end{pmatrix},$$
$$\mathbb{L}_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbb{L}_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbb{L}_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{33}$$

with decay rates γ_r, γ_g and γ_b. We can then solve the resulting Lindblad equation - the calculations will be in everything identical to the spin case. Considering an initial density matrix:

$$\rho_i = \begin{pmatrix} a(0) & b(0) & c(0) \\ b(0)^* & d(0) & f(0) \\ c(0)^* & f(0)^* & 1 - a(0) - d(0) \end{pmatrix} \tag{34}$$

The final expression for $\rho(t)$ is 3.2.

3.3 Splitting in the Position and Momentum Spaces - Restricted Fock Space Representation

Let $\mathcal{H}_1$ be the Hilbert space corresponding to the state of the particle before the splitting occurs, and $\mathcal{H}_2, \mathcal{H}_3$ the Hilbert spaces corresponding to the particles resulting from the splitting, we will consider the global Hilbert space as $\mathcal{H} = \mathcal{H}_1 \oplus S(\mathcal{H}_2 \otimes \mathcal{H}_3)$, where S symmetrizes appropriately (depending on the type of the particles involved) the space $\mathcal{H}_2 \otimes \mathcal{H}_3$. Thus, $\mathcal{H}$ is the linear subspace of the Fock space resulting from the restriction that we can only have one or two particles. Next, we will allow for new splitting phenomena to occur and take as the global Hilbert space the entire Fock space. In this section, we will be interested, primarily, in considering $\mathcal{H}_i = L^2(\mathbb{R}^3)$ — the space where we can treat the propagation of all the particles involved. Note that an element of this space is a pair $\mathbf{\Psi} = (\psi_1(\vec{x}_1), \psi_{2,3}(\vec{x}_2, \vec{x}_3))$ and the inner product in this space is given by $\langle(\psi_1, \psi_{2,3}), (\phi_1, \phi_{2,3})\rangle = \langle\psi_1, \phi_1\rangle_{\mathcal{H}_1} + \langle\psi_{2,3}, \phi_{2,3}\rangle_{S(\mathcal{H}_2 \otimes \mathcal{H}_3)}$. It is useful to start by studying how the relevant jump operators act on plane waves, as we will then assess how they act on wave packets. We define (for the case where the particles resulting from the splitting are distinguishable, i.e., when $S(\mathcal{H}_2 \otimes \mathcal{H}_3) = \mathcal{H}_2 \otimes \mathcal{H}_3$):

$$\left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle = (e^{i\vec{k}_1 \cdot \vec{x}_1}, 0) \qquad \left|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right\rangle = (0, e^{i\vec{k}_2 \cdot \vec{x}_2} e^{i\vec{k}_3 \cdot \vec{x}_3})$$

Each jump operator will be indexed by $\vec{k}'$, specifying how it acts on $\left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle$. Similarly to the cases in the previous sections, we consider that $L_{\vec{k}'}$ sends $\left|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right\rangle$ to 0.

$$L_{\vec{k}'}\left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle = \left|\mathbf{\Psi}_{2,\vec{k}';\vec{k}_1-\vec{k}'}\right\rangle \qquad L_{\vec{k}'}\left|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right\rangle = (0, 0) \tag{35}$$

For three non-relativistic particles, the Hamiltonian of the system is given by 36, and with this operator defined, the Lindblad equation is given by 37.

$$H_S = -\frac{\hbar^2}{2m_1}\nabla^2_{\vec{x}_1}\psi_1 \oplus \left(-\frac{\hbar^2}{2m_2}\nabla^2_{\vec{x}_2}\psi_2 \otimes \mathbb{1}_3 + \mathbb{1}_2 \otimes -\frac{\hbar^2}{2m_3}\nabla^2_{\vec{x}_3}\psi_3\right) \tag{36}$$

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_S, \rho] + \int_{\mathbb{R}^3} \gamma(\vec{k}')\left(L_{\vec{k}'}\rho L^\dagger_{\vec{k}'} - \frac{1}{2}\{L^\dagger_{\vec{k}'}L_{\vec{k}'}, \rho\}\right) d^3\vec{k}' \tag{37}$$

We denote by $\mathcal{D}[\rho]$ the dissipation term (the second term on the right-hand side of the Lindblad equation)

$$\rho(t) = \left(\begin{array}{ccc|ccc} e^{-\gamma_r t}a(0) & e^{-\frac{\gamma_r+\gamma_g}{2}}b(0) & e^{-\frac{\gamma_r+\gamma_b}{2}}c(0) & & & \\ e^{-\frac{\gamma_r+\gamma_g}{2}t}b(0)^* & e^{-\gamma_g t}d(0) & e^{-\frac{\gamma_g+\gamma_b}{2}t}f(0) & & \mathbf{0} & \\ e^{-\frac{\gamma_r+\gamma_b}{2}}c(0)^* & e^{-\frac{\gamma_g+\gamma_b}{2}t}f(0)^* & e^{-\gamma_b t}(1-a(0)-d(0)) & & & \\ \hline & \mathbf{0} & & (1-e^{-\gamma_r t})a(0) & 0 & 0 \\ & & & 0 & (1-e^{-\gamma_g t})d(0) & 0 \\ & & & 0 & 0 & (1-e^{-\gamma_b t})(1-a(0)-d(0)) \end{array}\right)$$

Figure 3. Final expression for $\rho(t)$ for partons with color.

and by $\rho_{1,\vec{k}_1,1,\vec{k}_1}(t)$ the quantity $\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho(t) \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle$. Let us compute, as a function of $\rho_{1,\vec{k}_1,1,\vec{k}_1}(t)$, the quantity $\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \mathcal{D}[\rho] \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle$:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \gamma(\vec{k}')\Big(\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| L_{\vec{k}'}\rho L_{\vec{k}'}^\dagger \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle - \\
&\qquad -\frac{1}{2}\Big(\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| L_{\vec{k}'}^\dagger L_{\vec{k}'}\rho \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle + \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho L_{\vec{k}'}^\dagger L_{\vec{k}'} \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle\Big)d^3\vec{k}' \\
&= -\frac{1}{2}\int_{\mathbb{R}^3} \gamma(\vec{k}')\Big(\left\langle \mathbf{\Psi}_{2,\vec{k}',\vec{k}_1-\vec{k}'}\right| L_{\vec{k}'}\rho \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle \\
&\qquad + \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho L_{\vec{k}'}^\dagger \left|\mathbf{\Psi}_{2,\vec{k}',\vec{k}_1-\vec{k}'}\right\rangle d^3\vec{k}' \\
&= -\frac{1}{2}\int_{\mathbb{R}^3} \gamma(\vec{k}')\Big(\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle + \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle\Big) d^3\vec{k}' \\
&= -\left(\int_{\mathbb{R}^3} \gamma(\vec{k}')d^3\vec{k}'\right) \rho_{1,\vec{k}_1,1,\vec{k}_1}(t)
\end{aligned} \tag{38}$$

Similarly for the quantity $\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| \mathcal{D}[\rho] \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle$, we obtain:

$$\begin{aligned}
&\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| \mathcal{D}[\rho] \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle \\
&= \int_{\mathbb{R}^3} \gamma(\vec{k}') \left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| L_{\vec{k}'}\rho L_{\vec{k}'}^\dagger \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle d^3\vec{k}' \\
&= \int_{\mathbb{R}^3} \gamma(\vec{k}') \left\langle \mathbf{\Psi}_{1,\vec{k}_2+\vec{k}_3}\right| \rho \left|\mathbf{\Psi}_{1,\vec{k}_2+\vec{k}_3}\right\rangle \delta(\vec{k}'-\vec{k}_2)d^3\vec{k}' \\
&= \gamma(\vec{k}_2)\, \rho_{1,\vec{k}_2+\vec{k}_3,1,\vec{k}_2+\vec{k}_3}(t)
\end{aligned} \tag{39}$$

The terms $\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| [H_S,\rho] \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle$ and $\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| [H_S,\rho] \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle$ are 0, since $\left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle$ and $\left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle$ are generalized eigenvectors of H_S. We denote by Γ the integral $\int_{\mathbb{R}^3} \gamma(\vec{k}')d^3\vec{k}'$. If we assume the splitting is isotropic, then $\Gamma = \int_{\mathbb{R}} \gamma(k')k'^2dk'$. By explicitly solving the equations, we obtain:

$$\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho(t) \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle = e^{-\Gamma t} \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho(0) \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle \tag{40}$$

$$\begin{aligned}
&\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| \rho(t) \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle = \\
&= \frac{\gamma(\vec{k}_2)}{\Gamma}(c - e^{-\Gamma t}) \left\langle \mathbf{\Psi}_{1,\vec{k}_2+\vec{k}_3}\right| \rho(0) \left|\mathbf{\Psi}_{1,\vec{k}_2+\vec{k}_3}\right\rangle
\end{aligned} \tag{41}$$

where c is an integration constant dependent on the initial condition. Similar to the previous sections, we assume that $\rho(0) = |\mathbf{\Psi}_0\rangle\langle\mathbf{\Psi}_0|$, with $|\mathbf{\Psi}_0\rangle = (\psi_1(\vec{x}_1), 0) \in \mathcal{H}$ (a pure state of a single particle), which implies that $c = 1$. We can verify that, in fact, $\operatorname{tr}\rho(t) = 1$:

$$\begin{aligned}
\operatorname{tr}\rho(t) &= \int_{\mathbb{R}^3} e^{-\Gamma t}\rho_{1,\vec{k}_1,1,\vec{k}_1}(0)\, d^3\vec{k}_1 \\
&\quad + \iint_{\mathbb{R}^6} \frac{\gamma(\vec{k}_2)}{\Gamma}(1-e^{-\Gamma t})\rho_{1,\vec{k}_2+\vec{k}_3,1,\vec{k}_2+\vec{k}_3}(0)\, d^3\vec{k}_3\, d^3\vec{k}_2
\end{aligned} \tag{42}$$

Let I be $\int_{\mathbb{R}^3} \rho_{1,\vec{k},1,\vec{k}}(0)\, d^3\vec{k}$, then (by making the change of variable $\vec{u} = \vec{k}_2 + \vec{k}_3$ in the second integral) we get

$$\operatorname{tr}\rho(t) = e^{-\Gamma t}I + (1-e^{-\Gamma t})I\frac{\int_{\mathbb{R}^3}\gamma(\vec{k}_2)\, d^3\vec{k}_2}{\Gamma} = I \tag{43}$$

but $I = 1$, due to the initial condition assumed, thus concluding the desired result. The function $\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho(t) \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle$ corresponds to the probability density of finding the system in a one-particle state with a given momentum. Explicitly, if we consider an initial state that is a wave packet: $|\mathbf{\Psi}_0\rangle = (\psi_1(\vec{x}_1), 0) = (\int_{\mathbb{R}^3} \phi(\vec{k}_1)e^{i\vec{k}_1\cdot\vec{x}_1}\, d^3\vec{k}_1, 0) \in \mathcal{H}$, the probability over time of finding the system in a one-particle state with momentum $\vec{k} \in \Omega$ is given by the integral:

$$\begin{aligned}
&\int_\Omega \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho(t) \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle\, d^3\vec{k}_1 = \\
&= e^{-\Gamma t}\int_\Omega \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| \rho(0) \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle\, d^3\vec{k}_1 \\
&= e^{-\Gamma t}\int_\Omega |\phi(\vec{k}_1)|^2\, d^3\vec{k}_1
\end{aligned} \tag{44}$$

The interpretation of $\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| \rho(t) \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle$ is analogous but for a two-particle state, now representing a joint probability density. Also, for $\rho(0) = |\mathbf{\Psi}_0\rangle\langle\mathbf{\Psi}_0|$ as defined earlier, the probability of finding the system in a two-particle state, the first with momentum $\vec{k}_2 \in \Omega_2$ and the second with momentum $\vec{k}_3 \in \Omega_3$, is given by:

$$\begin{aligned}
&\int_{\Omega_3}\int_{\Omega_2} \left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right| \rho(t) \left|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle\, d^3\vec{k}_2\, d^3\vec{k}_3 \\
&= \frac{(1-e^{-\Gamma t})}{\Gamma}\int_{\Omega_3}\int_{\Omega_2} \gamma(\vec{k}_2)|\phi(\vec{k}_2+\vec{k}_3)|^2\, d^3\vec{k}_2\, d^3\vec{k}_3
\end{aligned} \tag{45}$$

4 Interaction with a bath

4.1 Degenerate two level system interacting with a bath of harmonic oscillators

In this section, we'll study a degenerate two-level system interacting with a bath of harmonic oscillators. Once again, we consider the Hilbert space spanned by $\{|1\rangle, |2\rangle\}$ introduced in section 3.1. To reproduce the results of this section, we would like to create a model where $|1\rangle \to |2\rangle$ transitions occur with a certain probability. As a result, if the system starts in the $|1\rangle$ state, it converges exponentially to the $|2\rangle$ state, but now as a result of system-bath interaction. To do so, we consider the Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, that is the tensor product of the Hilbert spaces of the system and bath. The system Hamiltonian is as before: $H_S = \epsilon \mathbb{1}_S$. We model the thermal bath as a continuum of harmonic oscillators following [4], leading to the bath Hamiltonian:

$$H_B = \hbar \int_0^\infty a^\dagger(\omega) a(\omega) D(\omega) d\omega \tag{46}$$

where the operator a ($a^\dagger$) creates (anhililates) a *quanta* of energy $\hbar\omega$ in the bath and $D(\omega)$ is the bath density of states. Thus, the total (system + bath) Hamiltonian, acting on $\mathcal{H}$, is the operator:

$$\begin{aligned} H &= H_S + H_B + V \\ &= \epsilon \mathbb{1}_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes \hbar \int_0^\infty a^\dagger(\omega) a(\omega) D(\omega) d\omega + V \end{aligned} \tag{47}$$

where V is the interaction Hamiltonian between the system and the bath, which as detailed in section 2.3 can be decomposed as: $V = \sum_k A_k \otimes B_k$, $A_k^\dagger = A_k$ and $B_k^\dagger = B_k$. The Lindblad equation is then:

$$\begin{aligned} \frac{d}{dt}\rho_A(t) = &-\frac{i}{\hbar}\left[H_S + H_{LS}, \rho_A(t)\right] \\ &+ \sum_{k,\ell} \gamma_{k\ell}(0) \Big(A_\ell(0) \rho_A(t) A_k^\dagger(0) - \\ &\qquad - \tfrac{1}{2} \left\{ A_k^\dagger(0) A_\ell(0), \rho_A(t) \right\} \end{aligned} \tag{48}$$

As mentioned before, our system is intended to be degenerate. To accomplish this, we must force the H_{LS} (Lamb-Shift Hamiltonian) in equation 48 to be a multiple of the identity.

We already have seen that we can decompose H_{LS} as in 11. With this we also know the explicit form of $S_{kl}(\omega)$, 12, which automatically implies that $S_{kl}(\omega) \propto \mathbb{I}$. So, our study will be concentrated in the $\sum_\omega \sum_{k,\ell} A_k^\dagger(\omega) A_\ell(\omega)$ part, more precisely, the operators $A_k(\omega)$.

In our case, since the system only has two states with the same energy, the only possible value for ω is 0 and equation 9 reduces to:

$$A_k(\omega = 0) = \sum_{n,m=1,2} |n\rangle\langle n| A_k |m\rangle\langle m| = A_k \tag{49}$$

therefore, the relation that must be satisfied reduces to $\sum_{k,\ell} A_k^\dagger A_\ell \propto \mathbb{I}$ (and the jump operators in the Lindblad equation are the A_k).

Proposition 4.1. If we have $\{A_k\}$ as the set of operators that obey $A_k = x_k\sigma_x + y_k\sigma_y + z_k\sigma_z$ where $x_k, y_k, z_k \in \mathbb{R}$ ($\sigma_x, \sigma_y, \sigma_z$ are the Pauli Matrices), and a operator S which is simetric, $S = S^T$.Then, $\sum_{k,\ell}^n s_{kl} A_k^\dagger A_\ell \propto \mathbb{I}$ is true, where s_{kl} are the entries of the operator S.

Proof. In this proof, we will show that the squared terms and the cross terms are $\propto \mathbb{I}$ and thereby their sum must also be.

First, the Pauli Matrices are hermitian so, since all the coefficients are real, all possible A_k operators are hermitian:

$$\begin{aligned} A_k^\dagger &= A_k \\ \sum_{k,\ell}^n s_{kl} A_k^\dagger A_\ell &= \sum_{k,\ell}^n s_{kl} A_k A_\ell \end{aligned} \tag{50}$$

When expanding the sum we get:

$$\begin{aligned} &\sum_{k=1}^n \sum_{\ell=1}^n s_{kl} A_k A_\ell \\ &= s_{11}A_1A_1 + s_{12}A_1A_2 + \cdots + s_{21}A_2A_1 + \cdots + s_{nn}A_nA_n \\ &= \sum_{k=1}^n s_{kk}A_kA_k + \sum_{k>\ell}^n s_{kl}(A_kA_\ell + A_\ell A_k) \text{ , as } s_{kl} = s_{lk} \end{aligned} \tag{51}$$

Let's first look at the squared terms $s_{kk}A_kA_k$:

$$\begin{aligned} s_{kk}A_kA_k &= s_{kk}(x_k\sigma_x + y_k\sigma_y + z_k\sigma_z)(x_k\sigma_x + y_k\sigma_y + z_k\sigma_z) \\ &= s_{kk}[(x_k\sigma_x)^2 + ... + x_ky_x(\sigma_x\sigma_y + \sigma_y\sigma_x) + ...] \\ &= s_{kk}[(x_k^2 + y_k^2 + z_k^2)\mathbb{I} + 0] \propto \mathbb{I} \end{aligned} \tag{52}$$

Then, we analyse the cross terms $s_{kl}(A_kA_\ell + A_\ell A_k)$:

$$\begin{aligned} &s_{kl}(A_kA_\ell + A_\ell A_k) \\ &= s_{kl}[(x_k\sigma_x + y_k\sigma_y + z_k\sigma_z)(x_l\sigma_x + y_l\sigma_y + z_l\sigma_z) \\ &\quad + (x_l\sigma_x + y_l\sigma_y + z_l\sigma_z)(x_k\sigma_x + y_k\sigma_y + z_k\sigma_z)] \\ &= s_{kl}[2x_kx_l\sigma_x^2 + ... + x_ky_l(\sigma_x\sigma_y + \sigma_y\sigma_x) + ...] \\ &= s_{kl}[(2x_kx_l + 2y_ky_l + 2z_kz_l)\mathbb{I} + 0] \propto \mathbb{I} \end{aligned} \tag{53}$$

In our case we won't have to worry about the convergence of the sum, as it contains only finitely many terms. Since proportionality is preserved by the sum, we conclude that $\sum_{k,\ell}^n s_{kl} A_k^\dagger A_\ell \propto \mathbb{I}$.

□

We are now ready to properly begin building our model, we will take into account the result deduced in 4.1 to accomplish the degeneracy desired.

The first, and most important step is to define $V = \sum_k A_k \otimes B_k$. We will consider that k only goes to 1 for simplicity. We choose $A_1 = \sigma_x$, because we want this model to resemble and lead to a similar result to section 3.1, where we used the jump operator $L = |2\rangle\langle 1|$. Here $A_1 = |1\rangle\langle 2| + |2\rangle\langle 1|$ will have the same function.

Our B_1 will be the same as in the last example of section 6.2.5 of [4]:

$$B_1 = \int_{-\omega_{\max}}^{\omega_{\max}} B(\omega)\, d\omega, \text{with} \begin{cases} B(\omega) & = h(\omega) a_\omega, \\ B(-\omega) & = h(\omega) a_\omega^\dagger, \end{cases} \quad \text{for } \omega > 0 \tag{54}$$

where a_ω denotes the continuous bosonic operators $[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega')$, ω being the frequency of each bath mode.

Thereby we end up with the interaction term

$$\begin{aligned} V &= \sigma_x \otimes \int_0^{\omega_{\max}} h(\omega)(a_\omega^\dagger + a_\omega) d\omega \\ &= \int_0^{\omega_{\max}} (\sigma_x) h(\omega)(a_\omega^\dagger + a_\omega) d\omega \end{aligned} \tag{55}$$

We can now calculate the decay rate $\gamma_{k,\ell}(\omega)$ using equation 10. In this calculation we will use $\bar{n}(\omega) = [e^{(\hbar\omega/k_B T)} - 1]^{-1}$, that is the mean number of bosons of frequency ω in the thermal state ρ_b, given by the Bose-Einstein statistics; and a Ohmic spectral density of the bath $J(\omega) = h^2(\omega) = \eta\omega\theta(\omega_{max} - \omega)$ (this quantity defines the strength of the coupling per frequency). We calculate $\gamma_{k,\ell}$ in the degenerate case by taking the $\omega \to 0$ of equation 10.

$$\gamma_{11}(0) = \lim_{\omega_0 \to 0} 2\pi \ \mathrm{Tr}[B_1(\omega_0) B_1 \rho_b] \tag{56}$$

There seems to be a problem, as the limit does not exist. If $\omega_0 > 0$, we get $\gamma_{11} = \lim_{\omega_0 \to 0^+} 2\pi J(\omega_0)[\bar{n}(\omega_0) + 1]$ and if $\omega_0 > 0$, $\gamma_{11} = \lim_{\omega_0 \to 0^-} 2\pi J(\omega_0)\bar{n}(\omega_0)$. We can avoid this by considering a high temperature regime, where $\bar{n}(\omega) + 1 \approx \bar{n}(\omega)$. Thus we get,

$$\begin{aligned} \gamma_{11} &= \lim_{\omega_0 \to 0} 2\pi J(\omega_0)\bar{n}(\omega_0) \\ &= 2\pi \lim_{\omega_0 \to 0} \frac{\eta\omega_0\theta(\omega_{max} - \omega_0)}{e^{\omega_0/T} - 1}, \text{ using Cauchy Rule} \\ &= 2\pi \lim_{\omega_0 \to 0} \frac{\eta}{\frac{1}{T} e^{\omega_0/T}} = 2\pi\eta T \end{aligned} \tag{57}$$

Note: It's important to underline that the reason why we choose a Ohmic spectral density, was to have $J(\omega) \propto \omega$ since, combined with the fact that $\bar{n}(\omega) \propto 1/\omega$ for small ω, this choice ensures we obtain a finite limit. Other choices of spectral density would instead lead to the limit vanishing or diverging.

In our case, we don't need to obtain a simplified expression for $S_{11}(0)$ because, H_{LS} by construction, commutes with ρ_B and the initial term in the Lindblad equation is 0 independently of $S_{11}(0)$.[1]

Finally, we obtain the following Lindblad equation:

$$\begin{aligned} \frac{d}{dt}\tilde{\rho}_A(t) =& - i\,[H_{LS}, \tilde{\rho}_A(t)] + \\ & \sum_{k,\ell}^{1} \gamma_{k\ell}(0) \left(A_\ell(0)\, \tilde{\rho}_A(t)\, A_k^\dagger(0) - \tfrac{1}{2} \left\{ A_k^\dagger(0) A_\ell(0), \tilde{\rho}_A(t) \right\} \right) \\ =& - i[\sigma_x \sigma_x S_{11}(0), \tilde{\rho}_A(t)] \\ & + \gamma_{11}(0) \left(\sigma_x \tilde{\rho}_A(t) \sigma_x - \tfrac{1}{2} \{ \sigma_x \sigma_x, \tilde{\rho}_A(t) \} \right) \\ =& 0 + \gamma_{11}(0) \left(\sigma_x \tilde{\rho}_A(t) \sigma_x - \rho_A(t) \right) \end{aligned} \tag{58}$$

$$\begin{pmatrix} \frac{\partial a}{\partial t}(t) & \frac{\partial b}{\partial t}(t) \\ \frac{\partial b^*}{\partial t}(t) & -\frac{\partial a}{\partial t}(t) \end{pmatrix} = \gamma_{11}(0) \begin{pmatrix} d(t) - a(t) & -i\,\mathrm{Im}[b(t)] \\ i\,\mathrm{Im}[b(t)] & a(t) - d(t) \end{pmatrix} \tag{59}$$

with $\gamma_{11}(0) = 2\pi\eta T$. We obtain the coupled system:

$$\begin{cases} \frac{\partial a(t)}{\partial t} = \gamma\,(d(t) - a(t)), \\ \frac{\partial d(t)}{\partial t} = \gamma\,(a(t) - d(t)). \end{cases} \tag{60}$$

together with the equation: (noting that the equations for $b(t)$ and $b^*(t)$ are equivalent)

$$\frac{\partial b}{\partial t} = -\gamma(b(t) - b^*(t)) \tag{61}$$

which can be recast as a system of 2 equations by separating the real and imaginary part of $b(t) = x(t) + iy(t)$,

$$\begin{cases} \frac{\partial x(t)}{\partial t} = 0, \\ \frac{\partial y(t)}{\partial t} = -2\gamma y(t). \end{cases} \tag{62}$$

Both of these systems are easily solvable and we end up with the following density matrix,

$$\begin{aligned} \tilde{\rho}_A(t) &= \begin{pmatrix} a(t) & b(t) \\ b^*(t) & d(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(a_0 + d_0)(1 + e^{-2\gamma t}) & x_0 + y_0 e^{-2\gamma t} \\ x_0 - y_0 e^{-2\gamma t} & \frac{1}{2}(a_0 + d_0)(1 - e^{-2\gamma t}) \end{pmatrix} \end{aligned} \tag{63}$$

a_0, x_0, y_0, d_0 are the system initial conditions. If we impose that the system starts in a pure state $|1\rangle$ we obtain:

$$\tilde{\rho}_A(t) = \begin{pmatrix} \frac{1}{2}(1 + e^{-2\gamma t}) & 0 \\ 0 & \frac{1}{2}(1 - e^{-2\gamma t}) \end{pmatrix} \tag{64}$$

As we can see $\mathrm{Tr}[\tilde{\rho}_A(t)] = 1$ and $\lim_{t \to \infty} \tilde{\rho}_A(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, so we as time goes to infinity the system loses coherence and we end up with a classical randomness.

One interesting detail is that, as we know, the entropy $0 < S < log(N)$, where N is the dimension of the system A (in this case 2). Well, in this example, we go from a pure state where the entropy $S(0) = -\mathrm{Tr}[\tilde{\rho}_A log(t)(\tilde{\rho}_A(t))] = 0$ to the maximum entropy at infinity $lim_{t \to \infty} = -\mathrm{Tr}[\tilde{\rho}_A(t) log(\tilde{\rho}_A(t))] = -2(\frac{1}{2} log(\frac{1}{2})) = log(2)$ which shows the effect the bath has on the system.

[1] If we used a more complex interaction Hamiltonian, $V = \sum_k A_k \otimes B_k$, in other words if $k > 1$, we would need that all the elements of the set $\{B_k\}$ commuted with each other. So that the condition in 4.1 of $S = S^t$ would be obliged

4.2 Enforcing irreversibility

The derivation above results in a Lindblad equation with processes operators: one that leads to a transition from state $|1\rangle$ to state $|2\rangle$, that is, splits a particle in two, and one that induces the reverse transition, $|2\rangle$ to $|1\rangle$. The latter would correspond to two particles fusing into one, a process we do not consider in our phenomenological splitting model. Indeed, empirically we know that the splitting process in the LHC is irreversible, both for the vacuum and for the Quark Gluon Plasma. Thus, we are interested in finding a way to eliminate or at least disregard the this process operator.

One approach is to consider an effective non-hermitian interaction Hamiltonian:

$$V = \int_0^{\omega_{\max}} h(\omega)\sigma_-(a_\omega^\dagger + a_\omega)d\omega \tag{65}$$

We use $A_1 = \sigma_- = L$ (in the $\{|1\rangle, |2\rangle\}$ basis) and keep the B_1 used in section 4.1. Note that with this choice, we violate the requirement that H_{LS} is a multiple of the identity:

$$\begin{aligned} H_{LS} &= \sum_\omega \sum_{k,\ell} S_{k\ell}(\omega) A_k^\dagger(\omega) A_\ell(\omega) \\ &= S_{11}(0)\sigma_+\sigma_- = \begin{pmatrix} S_{11}(0) & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{66}$$

Since the decay rate only depends on the bath operator B_1, it will have the same value, $\gamma_{11}(0) = 2\pi\eta T$. Thus, we obtain a Lindblad equation similar to that of section 3.1, with an extra term due to H_{LS}:

$$\begin{aligned} \frac{d}{dt}\rho_A(t) &= -\frac{i}{\hbar}\left[H + H_{LS}, \tilde{\rho}_A(t)\right] \\ &\quad + \sum_{k,\ell}^{1} \gamma_{k\ell}(0)\Big(A_\ell(0)\,\rho_A(t)\,A_k^\dagger(0) - \\ &\qquad - \tfrac{1}{2}\left\{A_k^\dagger(0)A_\ell(0), \rho_A(t)\right\}\Big) \\ &= -\frac{i}{\hbar}[S_{11}(0)L^\dagger L, \rho_A(t)] \\ &\quad + \gamma_{11}(0)\left(L\rho_A(t)L^\dagger - \tfrac{1}{2}\{L^\dagger L, \rho_A(t)\}\right) \end{aligned} \tag{67}$$

This equation describes a unidirectional process as we desired, but introduces a Lamb-shift of $S_{11}(0)$ between the 2 levels. If we consider once again an Ohmic spectral density, $J(\omega) = h^2(\omega) = \eta\omega\theta(\omega_{max} - \omega)$, it is possible to calculate this shift using equation 12:

$$\begin{aligned} S_{11}(0) &= -\text{P.V.}\int_{-\omega_{\max}}^{\omega_{\max}} d\omega' \frac{\text{Tr}\left[B_1(\omega')B_1\rho_B\right]}{\omega'} \\ &= -\text{P.V.}\int_0^{\omega_{\max}} d\omega' \frac{\text{Tr}\left[B_k(\omega')B_\ell\rho_B\right]}{\omega'} \\ &\quad + \text{P.V.}\int_0^{\omega_{\max}} d\omega' \frac{\text{Tr}\left[B_k(-\omega')B_\ell\rho_B\right]}{\omega'} \\ &= \frac{1}{4}\text{P.V.}\int_0^{\omega_{\max}} d\omega' J(\omega')\left[-\frac{\bar{n}(\omega')+1}{\omega'} + \frac{\bar{n}(\omega')}{\omega'}\right] \\ &= -\frac{1}{4}\text{P.V.}\int_0^{\omega_{\max}} \eta\frac{\omega'}{\omega'}d\omega' = -\frac{\omega_{max}}{4} \end{aligned} \tag{68}$$

The new term in the Lindblad equation is equal to

$$\begin{aligned} -\frac{i}{\hbar}[S_{11}(0)L^\dagger L, \rho_A(t)] &= -\frac{i}{\hbar}\left[\begin{pmatrix} S_{11}(0) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}\right] \\ &= -\frac{i}{\hbar}\begin{pmatrix} 0 & S_{11}(0)b \\ -S_{11}(0)b^* & 0 \end{pmatrix} \end{aligned} \tag{69}$$

Thus, the Lindblad equation reduces to a form similar to equation 17, but with this new term:

$$\begin{aligned} \frac{\partial\rho}{\partial t} &\equiv \begin{pmatrix} \frac{\partial a}{\partial t}(t) & \frac{\partial b}{\partial t}(t) \\ \frac{\partial b^*}{\partial t}(t) & -\frac{\partial a}{\partial t}(t) \end{pmatrix} \\ &= \frac{i}{\hbar}\begin{pmatrix} 0 & -S_{11}(0)b(t) \\ S_{11}(0)b(t)^* & 0 \end{pmatrix} + \gamma\begin{pmatrix} -a(t) & -\frac{b(t)}{2} \\ -\frac{b(t)^*}{2} & a(t) \end{pmatrix} \end{aligned} \tag{70}$$

The equation for $a(t)$ remains unchanged, whereas the equation for $b(t)$ is now $\dot{b}(t) = -\left(\frac{\gamma}{2} - \frac{i}{\hbar}S_{11}(0)\right)b(t)$. Nevertheless, with our usual initial condition, $\rho(0) = |1\rangle\langle 1|$, this change is irrelevant and we obtain once again the time evolution of equation 18, reproducing the results of section 3.1.

4.3 Bath Interaction in the Position and Momentum Spaces

We now generalize the previous discussion to the continuous Hilbert space of position and momentum in order to describe the splitting of particles moving in a 3D space due to system-bath interaction, in the same way as section 3.3 generalized section 3.1. Using once again the $L_{\vec{k}}$ operators defined in section 3.3, we consider an interaction Hamiltonian of the form:

$$V = \iint g(\omega, \vec{k})(L_{\vec{k}} + L_{\vec{k}}^\dagger)(a_\omega^\dagger + a_\omega)d\omega d^3\vec{k} \tag{71}$$

This Hamiltonian has a clear physical interpretation: each time a splitting (which can have any $\vec{k}$ for the momentum difference) occurs in the system, it is because a quanta was absorbed by the system from the bath or emitted by the system into the bath.

For it to be possible to factorize V in terms of hermitian operators, we must have that $g(\omega, \vec{k}) = h(\omega)f(\vec{k})$ and then $V = A \otimes B$, $A = \int f(\vec{k})(L_{\vec{k}} + L_{\vec{k}}^\dagger)d^3\vec{k}$ and $B = \int h(\omega)(a_\omega^\dagger + a_\omega)d\omega$

Using a generalization of equation 9 for a continuous system Hilbert space, we can calculate the jump operators:

$$
\begin{aligned}
&A(\omega) = \\
&= \int \left(\left|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right\rangle\!\left\langle \mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right| A \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle\!\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| + h.c. \right) \\
&\qquad \delta\left(\frac{k_2^2}{2m_2} + \frac{k_3^2}{2m_3} - \frac{k_1^2}{2m_1} - \omega \right) d^3\vec{k}_1 d^3\vec{k}_2 d^3\vec{k}_3 \\
&= \int f(\vec{k}) \left(\left|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right\rangle\!\left\langle \mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right| L_{\vec{k}} \left|\mathbf{\Psi}_{1,\vec{k}_1}\right\rangle\!\left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| + h.c. \right) \\
&\qquad \delta\left(\frac{k_2^2}{2m_2} + \frac{k_3^2}{2m_3} - \frac{k_1^2}{2m_1} - \omega \right) d^3\vec{k}_1 d^3\vec{k}_2 d^3\vec{k}_3 d^3\vec{k} \\
&= \int f(\vec{k}) \left(\left|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\right\rangle \delta\left(\vec{k_2} - \vec{k}\right) \delta\left(\vec{k_3} - \vec{k_2} + \vec{k}\right) \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| + h.c. \right) \\
&\qquad \delta\left(\frac{k_2^2}{2m_2} + \frac{k_3^2}{2m_3} - \frac{k_1^2}{2m_1} - \omega \right) d^3\vec{k}_1 d^3\vec{k}_2 d^3\vec{k}_3 d^3\vec{k} \\
&= \int f(\vec{k}) \underbrace{\left(\left|\mathbf{\Psi}_{2,\vec{k};\vec{k}_1-\vec{k}}\right\rangle \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right| + h.c. \right)}_{\mathcal{A}_{\vec{k}_1,\vec{k}}} \\
&\qquad \cdot \delta\left(\underbrace{\frac{k^2}{2m_2} + \frac{(\vec{k}_1 - \vec{k})^2}{2m_3} - \frac{k_1^2}{2m_1}}_{\Delta(\vec{k}_1,\vec{k})} - \omega \right) d^3\vec{k}_1 d^3\vec{k}
\end{aligned}
\tag{72}
$$

The physical interpretation of these jump operators is clear: $A(\omega)$ corresponds to the splitting of a particle where the energy difference between the final two-particle state and the initial one-particle state is $\hbar\omega$. Using equation 10, we can calculate the decay rates (considering $\omega > 0$):

$$
\begin{aligned}
\gamma(\omega) &= 2\pi \mathrm{Tr}[B_1(\omega) B_1 \rho_B] \\
&= \frac{\pi}{2} h(\omega) \int_0^{\omega_{\max}} d\omega' h(\omega') \mathrm{Tr}[a_\omega (a_{\omega'} + a_{\omega'}^\dagger) \rho_B] \\
&= \frac{\pi}{2} h^2(\omega) \mathrm{Tr}[a_\omega a_\omega^\dagger \rho_B] \\
&= \frac{\pi}{2} h^2(\omega) [\bar{n}(\omega) + 1] \\
\gamma(-\omega) &= 2\pi \mathrm{Tr}[B_1(-\omega) B_1 \rho_B] \\
&= \frac{\pi}{2} h(\omega) \int_0^{\omega_{\max}} d\omega' h(\omega') \mathrm{Tr}[a_\omega^\dagger (a_{\omega'} + a_{\omega'}^\dagger) \rho_B] \\
&= \frac{\pi}{2} h^2(\omega) \mathrm{Tr}[a_\omega^\dagger a_\omega \rho_B] \\
&= \frac{\pi}{2} h^2(\omega) \bar{n}(\omega)
\end{aligned}
\tag{73}
$$

This results in the Lindblad equation:

$$
\begin{aligned}
\frac{d}{dt}\rho(t) = &-\frac{i}{\hbar} [H + H_{LS}, \rho(t)] + \\
&+ \underbrace{\int_{-\infty}^{+\infty} \gamma(\omega) \left(A(\omega) \rho(t) A^\dagger(\omega) - \tfrac{1}{2} \left\{ A^\dagger(\omega) A(\omega), \rho(t) \right\} \right) d\omega}_{D[\rho]}
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
D[\rho] = &\iint f^2(\vec{k}) \left(\mathcal{A}_{\vec{k}_1,\vec{k}} \rho(t) \mathcal{A}_{\vec{k}_1,\vec{k}}^\dagger - \tfrac{1}{2} \left\{ \mathcal{A}_{\vec{k}_1,\vec{k}}^\dagger \mathcal{A}_{\vec{k}_1,\vec{k}}, \rho(t) \right\} \right) \\
&\left(\int_{-\infty}^{+\infty} \frac{\pi}{2} h^2(\omega) \bar{n}(\omega) \delta\left(\Delta(\vec{k}_1, \vec{k}) - \omega \right) d\omega + \right. \\
&\left. + \int_0^{+\infty} \frac{\pi}{2} h^2(\omega) \delta\left(\Delta(\vec{k}_1, \vec{k}) - \omega \right) d\omega \right) d^3\vec{k}_1 \, d^3\vec{k}
\end{aligned}
$$

Performing the integral over ω, we obtain:

$$
\begin{aligned}
D[\rho] = &\iint \frac{\pi}{2} g^2(\Delta(\vec{k}_1, \vec{k}), \vec{k}) \left(\bar{n}(\Delta(\vec{k}_1, \vec{k})) + \frac{1}{2} \right) \\
&\left(\mathcal{A}_{\vec{k}_1,\vec{k}} \rho(t) \mathcal{A}_{\vec{k}_1,\vec{k}}^\dagger - \tfrac{1}{2} \left\{ \mathcal{A}_{\vec{k}_1,\vec{k}}^\dagger \mathcal{A}_{\vec{k}_1,\vec{k}}, \rho(t) \right\} \right) d^3\vec{k}_1 \, d^3\vec{k}
\end{aligned}
\tag{75}
$$

Identifying $\int \mathcal{A}_{\vec{k}_1,\vec{k}} d^3\vec{k}_1 = L_{\vec{k}} + L_{\vec{k}}^\dagger$, we obtain an equation similar to 37 of section 3.3, but with $L_{\vec{k}} + L_{\vec{k}}^\dagger$ as a jump operator rather than $L_{\vec{k}}$. Alternatively, we can see this equation as a continuous version of equation58 of section 4.1 since $L + L^\dagger = \sigma_x$. Hence, we once again obtain a reversible splitting process. To impose irreversibility, we could use a non-hermitian Hamiltonian as in section 4.2:

$$
V = \iint g(\omega, \vec{k}) L_{\vec{k}} (a_\omega^\dagger + a_\omega) d\omega d^3\vec{k} \tag{76}
$$

which would lead to the Lindblad equation:

$$
\frac{d}{dt}\rho(t) = -\frac{i}{\hbar} [H + H_{LS}, \rho(t)] + D[\rho] \tag{77}
$$

$$
\begin{aligned}
D[\rho] = &\iint \frac{\pi}{2} g^2(\Delta(\vec{k}_1, \vec{k}), \vec{k}) \left(\bar{n}(\Delta(\vec{k}_1, \vec{k})) + \frac{1}{2} \right) \\
&\left(\mathcal{L}_{\vec{k}_1,\vec{k}} \rho(t) \mathcal{L}_{\vec{k}_1,\vec{k}}^\dagger - \tfrac{1}{2} \left\{ \mathcal{L}_{\vec{k}_1,\vec{k}}^\dagger \mathcal{L}_{\vec{k}_1,\vec{k}}, \rho(t) \right\} \right) d^3\vec{k}_1 \, d^3\vec{k}
\end{aligned}
\tag{78}
$$

with $\mathcal{L}_{\vec{k}_1,\vec{k}} = \left|\mathbf{\Psi}_{2,\vec{k};\vec{k}_1-\vec{k}}\right\rangle \left\langle \mathbf{\Psi}_{1,\vec{k}_1}\right|$. Comparing this to equation 37 and the results of section 3.3, we can identify the decay rate:

$$
\begin{aligned}
\gamma(\vec{k}_1, \vec{k}) &= \frac{\pi}{2} g^2(\Delta(\vec{k}_1, \vec{k}), \vec{k}) \left(\bar{n}(\Delta(\vec{k}_1, \vec{k}) + \frac{1}{2} \right) \\
\Gamma &= \iint \gamma(\vec{k}_1, \vec{k}) d^3\vec{k}_1 \, d^3\vec{k}
\end{aligned}
\tag{79}
$$

4.4 Vacuum and Quark Gluon Plasma

The previous results can be used to describe the splitting phenomenon due to system-bath interaction irrespective of the medium in which the partons are propagating. Namely, they apply to the case where splitting occurs as a result of vacuum fluctuations, in which case the medium is the vacuum, and to the case where the quark gluon plasma, which affects splitting, is present. The difference between these two cases will be mainly contained in the γ factors, since these depend on the form of the interaction between the system (the partons who undergo splitting) and the bath

(the medium). In the vacuum case, the Lindblad equation is given by:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_S, \rho] + \int_{\mathbb{R}^3} \gamma_{\text{vac}}(\vec{k}') \left(L_{\vec{k}'} \rho L_{\vec{k}'}^\dagger - \frac{1}{2}\{L_{\vec{k}'}^\dagger L_{\vec{k}'}, \rho\} \right) d^3\vec{k}' \tag{80}$$

In the QGP case, we need to sum new terms:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_S, \rho] + \int_{\mathbb{R}^3} \gamma_{\text{vac}}(\vec{k}') \left(L_{\vec{k}'} \rho L_{\vec{k}'}^\dagger - \frac{1}{2}\{L_{\vec{k}'}^\dagger L_{\vec{k}'}, \rho\} \right) + \\ + \gamma_{\text{QGP}}(\vec{k}') \left(L_{\vec{k}'} \rho L_{\vec{k}'}^\dagger - \frac{1}{2}\{L_{\vec{k}'}^\dagger L_{\vec{k}'}, \rho\} \right) d^3\vec{k}' \tag{81}$$

Note that the vacuum terms remain since vacuum fluctuations are always present.

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