LIP-STUDENTS-20-15

From light-front wave functions to parton distribution functions

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Abstract. In this work the possibility of obtaining Parton Distribution Functions (PDFs) from the calculation of Light-Front Wave Functions is explored. Firstly, a brief theoretical explanation is given, followed by a review of the method for calculating Light-Front Wave Functions (LFWFs) using the Nakanishi Weight Function (presented by Federico et al.), which will be used as a baseline to compare future results. The numerical methods and challenges are also discussd. A different method for the calculation of LFWFs based on the direct calculation from the Bethe-Salpeter Equation (BSE) is proposed.

KEYWORDS: QCD, Bethe-Salpeter Equation, Light-Front Wave Functions, Parton Distribution Functions

1 Introduction

Quantum Chromodynamics is the SU(3) gauge theory concerned with the description of the strong interaction force. QCD adds quarks and gluons (similar to the electrons and photons of quantum electrodynamics) to the mix of known elementary physical particles.

Quarks are spin-$\frac{1}{2}$ fermions that carry what is called a color (hence *Chromodynamics*) charge, commonly referred to as red, green or blue. There are six flavours of them. Gluons are the spin-1 bosonic force carriers, that also have color charge, which can be thought of as a combination of the previous three (8 possibilities).

Compared with othe the interactions, like QED, QCD has a special distinguishing feature: asymptotic freedom. As such, at close distances (that is, high momenta), particles behave as they were free, with the attraction between particles increasing with the distance of separation - the so called confinement.

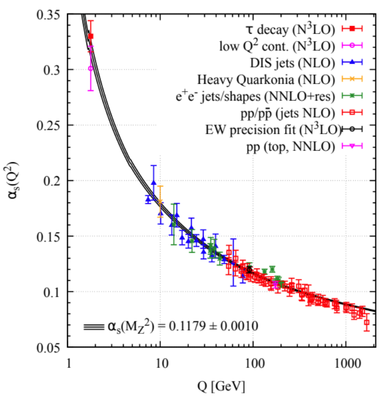

Figure 1. Experimental measures of the coupling constant $\alpha_S(Q^2)$, as a function of the energy scale Q, taken from P. A. Zyla et al., Prog. Theor. Exp. Phys. 2020, 083C01 (2020)

This confinement reveals another important constraint: all physical detectable particles are color-neutral, that is, one can only find quarks in bound color-neutral states. Therefore it is impossible to detect free quarks (or any other color-charged combination).

This characteristic imposes a severe restriction when doing perturbative calculations in QCD: at low momenta (short distances), perturbative methods are not valid, and as such, one needs to find alternative methods to make predictions [1].

1.1 Baryons

Baryons are compound objects made of a colorless combination of three quarks and of their interactions via gluons. These are among the most important particles in the known universe, as they include neutrons and protons - the building blocks of visible matter.

While it is tempting to look at them as composed of three separate quarks, interacting in a mean-field type potential (think about the coulomb potential, for example), this picture could not be farther from the truth.

QCD is a complex theory, and the idea of three lonely quarks with simple interactions is challenged by the continuous creation and annihilation of quark-antiquark pairs ($q\bar{q}$) and very strong gluon-gluon effects.

In reality, a much more suited image is thinking about a bag filled with quarks and $q\bar{q}$ pairs (also called *partons*), which happens to have three valence particles responsible for its properties. The proton, for example, has *uud* valence quarks.

These objects are non-perturbative, and as such, one needs to work with the full QCD theory in order to be able to make predictions about their behaviour. [2] [3].

1.2 Parton Distribution Functions (PDFs)

One of the objectives of studying baryons is to determine their structure. As such, there is interest in finding out how the baryon's total momentum is distrubuted among the components (*partons*).

That is the definition of the Parton Distribution Function. These functions $f_i(x, Q^2)$ give the probability that the species i has a fraction x of the total momentum Q^2.

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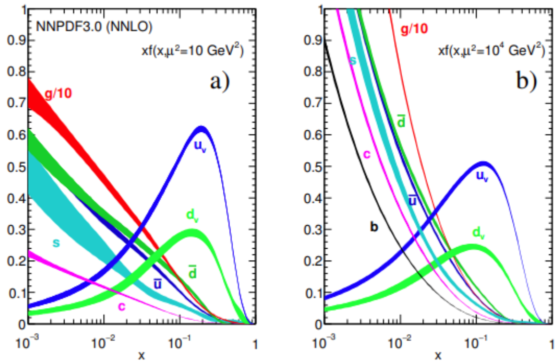

Figure 2. Proton Parton Distribution Functions at two different energy scales $Q^2 = 10\text{GeV}^2$ (a) and $Q^2 = 10^4\text{GeV}^2$. Taken from [4]

These functions are very interesting as they are universal (e.g every proton shares the same PDF) and used in the calculation of several other experimental and theoretical quantities such as cross-sections (eg. Drell-Yan, Deep Inelastic Scattering), experimental background among others.

Since PDFs are non perturbative, there is no straightfoward way to calculate them from the theory. As a result, our current understanding of PDFs comes from the fitting of experimental results. Figure 2 shows a plot of the proton PDFS, at different energy scales. Note the presence of some species like c and s quarks not predicted by the simple quark model.

2 Calculating PDFs

This work is focused in the calculations of the PDFs directly from the theory, rather than relying on experimental results.

As a starting point, a simple model will be studied. This model includes a scalar particle ϕ (of mass m), and a scalar interaction χ (of mass μ), with a simple interacting Lagrangian $\mathcal{L} = g\phi\phi\chi$.

The plan is as follows:

1. Calculate the solutions of the Bethe-Salpeter equation, directly and via the Nakanishi representation [5].
2. Calculate the Light-Front Wave functions, directly and via the Nakanishi representation [5].
3. Calculate PDFs via the obtained LFWFs. (See [6] for different methods).

2.1 Bethe-Salpeter Equation

In non-relativistic Quantum Mechanics, there is a fairly straightfoward way to get bound states from a theory. We just need to solve the Schrödinger Equation for that specific system: $H\psi = E\psi$.

In Quantum Field Theory, however, the fundamental quantities are n-point correlation functions, that is, objects with the following structure, built from the theory's vacuum $|0\rangle$ and its fields $\phi(x_i)$:

$$G(x_1, x_2, x_3, ...) = \langle 0|\, \mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)...\,|0\rangle \quad (1)$$

Bound states are extracted from these correlation functions via the fact that bound states $|\lambda\rangle$ with on-shell momentum $p^2 = m_\lambda^2$, produce a pole in the correlation function (Fourier transformed to momentum space).

The Bethe-Salpeter wave function $\Psi(\{x_i\})$ is then defined as the residue of these poles (see [7] for more detailed calculations). For example, a two-quark BSWF is given by

$$\Psi(x, P) = \langle 0|\, \mathrm{T}\phi(0)\phi(x)\,|\lambda(P)\rangle\,, \quad (2)$$
$$\Psi(k, P) = \int d^4x e^{-ik\cdot x}\Psi(x, P). \quad (3)$$

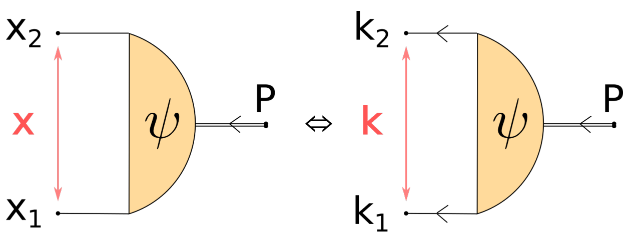

Figure 3. Schematical view of a Bethe-Salpeter Wave Function

These Ψ are calculated via the Bethe-Salpeter Equation, which takes the form $\Psi = G_0 K \Psi$, where K is built from all irreducible diagrams in the theory and G_0 is the propagator of the particles. [2]. In the following figure 4, a schematic view is shown.

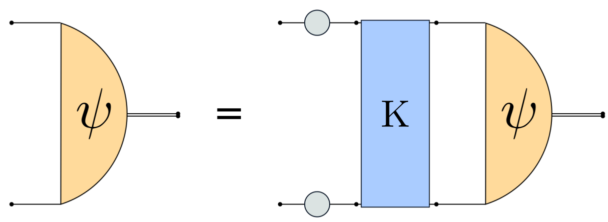

Figure 4. Schematical view of the Bethe-Salpeter Equation

One can write the $\Psi = G_0\psi$, where small ψ is the Bethe-Salpeter Amplitude. The Bethe-Salpeter Equation can be transformed to an equation for the amplitude by [8].

$$\psi(k, P) = \int \frac{d^4k}{(2\pi)^4} K(k, q) G_0(q, P)\psi(q, P). \quad (4)$$

2.2 Nakanishi Representation

It is possible to define a new function - the Nakanishi weight function - that depends on two real variables and is non-0singular, so that our BSWF can be written as an integral in momentum space of this weight function and a factor that contains the analytical structure [5]

Going to a Euclidian metric (instead of the Minkowski variables in [5]), where x is the radial component and z is the angular dependency of the momentum, one can write the definition of the Nakanishi weight function $h(x,z)$ for the simple scalar model with $t = \frac{P^2}{4m^2}$:

$$\psi(x,z) = \frac{1}{m^4}\int_0^\infty dx' \int_{-1}^1 dz' \frac{(1-z'^2)h(x',z')}{\left[x + x' + 1 + t + \sqrt{xt}zz'\right]^3}. \tag{5}$$

In the same way, by substitution, one can reach the Bethe-Salpeter Equation, but this time written for the Nakanishi Function $h(x,z)$, as shown in eqs. 16, 17, 18, 19 below.

This is an integral equation, with the added difficulty of having integrals on both sides. To try and solve this equations numerically, a basis for z is adopted, as defined in [5], using the C_i^α - the Gegenbauer polynomials:

$$(1-z^2)h(x,z) = g(x,z) = \sum_m h_m(x)Y_m(z), \tag{6}$$

$$Y_m(z) = 4(1-z^2)\Gamma\left(\frac{5}{2}\right)\sqrt{\frac{\left(2l+\frac{5}{2}\right)(2l)!}{\pi\Gamma\left(\frac{5}{2}\right)}}C_{2m}^{\frac{5}{2}}(z), \tag{7}$$

$$\int_{-1}^1 dz Y_m(z)Y_{m'}(z) = \delta_{mm'}. \tag{8}$$

After discretizing in z, a more condensed version of the BSE for the Nakanishi weight can be found:

$$\int dx' B_{mn}(x,x')g_n(x') = \int dx' K_{mn}(x,x')g_n(x'). \tag{9}$$

Discretizing the x direction as well, the equations can now be written as a simple eigenvalue equation, provided that the **B** operator is invertible. To help with that, a small diagonal parameter $\epsilon\delta_{ij}$ was added for regularization. Without this parameter, the numerics are only stable for very small matrix sizes. The equation to solve is:

$$\lambda\mathbf{B}\mathbf{h} = \mathbf{K}\mathbf{h} \implies \mathbf{B}^{-1}\mathbf{K}\mathbf{h} = \lambda\mathbf{h}. \tag{10}$$

2.3 Light-Front Wave Functions

The next step is now to transform the results for the BSWF to the Light Front Dynamics. In this case, we are dealing with all distances where $x^2 = 0$, that is, light-like.

For that, a light-like vector n is defined as $n = (0,0,1,-i)^T$, such that x can be written as $x = \lambda n$. It is immediate that $n^2 = x^2 = 0$.

For the momentum 4-vector, a quantity $k^+ = k \cdot n$ is defined that accounts for the momentum along the x direction.

A variable ξ can then be defined that gives the fraction of the total momentum $P = k_1 + k_2$ which is carried by the particle with momentum k_1^+. Naturally, $\xi \in (0,1)$. Applying to the rest of the momentum variables:

$$k_1^+ = \xi P^+, \tag{11}$$

$$k_2^+ = (1-\xi)P^+, \tag{12}$$

$$k = \frac{k_1 - k_2}{2} \implies k^+ = \left(\xi - \frac{1}{2}\right)P^+ = -\frac{\alpha}{2}P^+. \tag{13}$$

The variable $\alpha \in (-1,1)$, defined in eq. 13 will be the argument of the Light-Front Wave Function $\psi(\alpha)$, which can be directly defined via the BSWF. For details on how to do this with the Nakanishi Representation, see [5] and [9]. Beginning with the Fourier transform of the BSWF, it is possible to write $\psi(\alpha)$ as:

$$\psi(\alpha) = \int \frac{d\lambda}{4\pi} e^{i\frac{\alpha}{2}P\cdot n\lambda}\psi(\lambda n, P) \tag{14}$$

$$\psi(\alpha) = \int \frac{d^4k}{(2\pi)^4}\int \frac{d\lambda}{4\pi} e^{i\lambda(k\cdot n + \frac{\alpha}{2}P\cdot n)}\psi(k,P) \tag{15}$$

The integral over λ will give a Dirac Delta function, with a condition on momentum: $\frac{1}{2}\delta(k\cdot n + \frac{\alpha}{2}P\cdot n)$. Combined with the integration on k ($d^4k = \frac{1}{2}d^2k_\perp dk^+dk^-$), this defines $\psi(\alpha)$ as an averaging on $k_\perp$ (perpendicular to the x direction), on k^- and evaluating the BSWF when $k^+ = -\frac{\alpha}{2}P^+$.

3 Results

3.1 Bethe-Salpeter Wave Function

The Bethe-Salpeter Wave Function was obtained with both methods: using the Nakanishi Representation, as shown in [5] and in section 2.2; and directly solving the Bethe-Salpeter Amplitude, as in eq. 4.

Both are integral equations that can be reduced to the form of an eigenvalue problem. For both the power method was used, that is ($\psi_{i+1} = \frac{1}{\lambda_i}\mathbf{M}\psi_i$), which converges to the ground state. In principle, with more advanced numerical solvers, one could retrieve more eigenvalues and eigenvectors from the equations.

The results presented were calculated using $\sqrt{t} = 0.75i$, and a $\beta = \frac{\mu}{m} = 4$, and can be seen in figure 5. See [8] for details.

One particular result that is immediate is that the BS amplitude is pratically independent of the angular coordinate (z). The only dependance is in the radial coordinate x. Note that the x coordinate is in logarithmic scale. As a simple model, one could approximate the amplitude e.g. by $\frac{1}{1+\gamma}$, where γ is a real parameter.

$$\int_0^\infty dx' \frac{h(x',z)}{[x' + \mathcal{N}(x,z)]^2} = \int_0^\infty dx' \int_{-1}^{1} dz' V(x,z,z',z') \frac{1-z'^2}{1-z^2} h(x',z') \tag{16}$$

$$V(x,x',z,z') = \frac{g^2}{(4\pi m)^2} \frac{1}{\mathcal{N}(x,z)} \int_0^1 dv \left[K(v,x,x',z,z') + K(v,x,x',-z,-z')\right] \tag{17}$$

$$K(v,x,x',z,z') = \frac{\theta(z'-z)(1+z)^2 v^2}{\left[v(1-v)(1+z')\mathcal{N}(x,z) + v^2(1+z)\mathcal{N}(x',z') + (1-v)(1+z)(\frac{\mu}{m} + vx')\right]^2} \tag{18}$$

$$\mathcal{N}(x,z) = x + 1 + t(1-z^2) \tag{19}$$

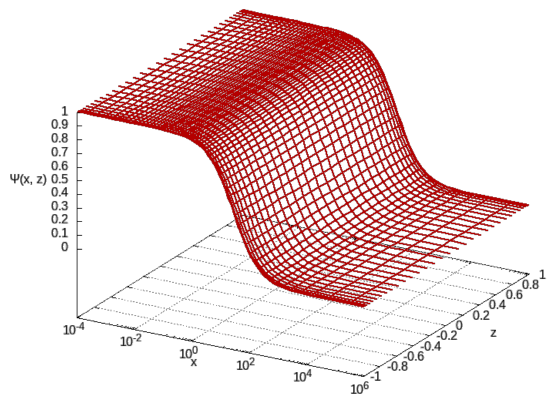

Figure 5. Bethe-Salpeter Amplitude obtained via directly solving eq. 4

As a comparison, a plot combining the Nakanishi result and the direct result for the BSWF is presented. The BSWF for the direct case was obtained by mutliplying the calculated amplitude with the propagators $\frac{1}{(x+t+1)^2-4xtz^2}$ (see [8]).

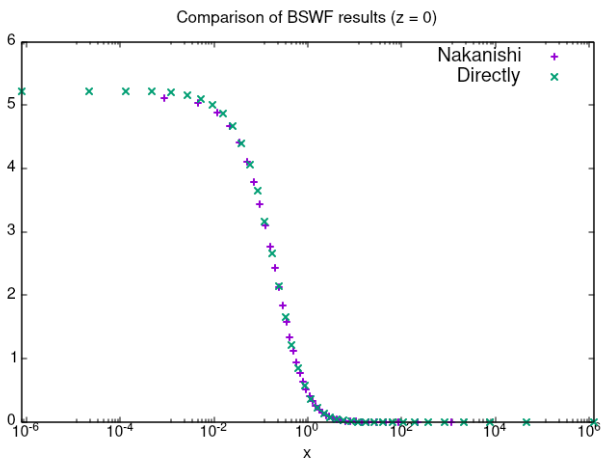

Figure 6. A comparison between the BSWF results for the Nakanishi Representation and the direct integration of the BSE for the Amplitude

The results obtained with both methods are in agreement. There is one thing to note however: the Nakanishi curve has less sampling points. This is resulting from the fact that this method is more computationally expensive as each entry of the operators defined in eq. 10 requires the solution of at least two integrals and a futher expansion in the z basis.

3.2 Light-Front Wave Functions

The results for the Light-Front Wave Functions are now presented.

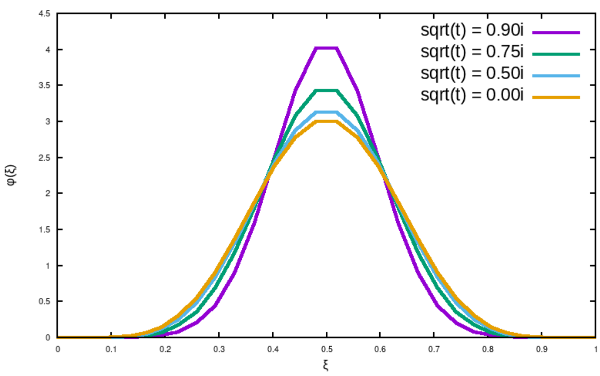

Figure 7. Light-Front Wave Functions results obtained with $\beta = 0.50$, for various values of $\sqrt{t}$. Note that the variable ξ is used instead of the α defined in eq. 14. This is in line with the notation on [5]

The results show that the distribution on ξ have a very smooth shape, peaked at the $\xi = 0.5$ point. This means that in this simple model, the most likely configuration is both particles sharing the same fraction of total momentum.

The curves are symmetric around the $\xi = 0.5$ axis, which was expected: both particles are identical and indistinguishable.

4 Further Work

The next step is the direct integration of the BSWF as explained in section 2.3.

There are still details about how to proceed numerically with the δ-function that are yet not fully understood and solved. This will be the focus of future work.

So far, a simple approximation has been used for the BSWF, based on the results obtained previously on section 3.

$$\Psi(x,z) \approx \frac{1}{(x+t+1)^2 - 4xtz^2}\frac{1}{x+\gamma} \tag{20}$$

Intermediate attempts have resulted in curves that are non symmetric and others more similar to figure 7, that, however, fail to vanish at the $|\alpha| = 1$ endpoints.

There is also one added difficulty: for $\sqrt{t}$ approaching the imaginary axis, the singularities in the propagator form branch cuts after integrating on z that create forbidden zones. The integration on x must be through a deformed contour that allows the avoidance of these cuts. It appears that using the non-singular Nakanishi Weight Function trades more numerical cost for a much simpler analytical structure.

There has already been an application of these contour deformations on the BSE, as seen in [8].

Future plans for this work are to explore the structure of the singularities so that one suitable formulation and path can be found, that results in curves with the desired characteristics and in accordance to the calculations done via the Nakanishi Representation. Afterwards, there remains the actual calculation of the PDFs and perhaps possible application to more complex Lagrangians.

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