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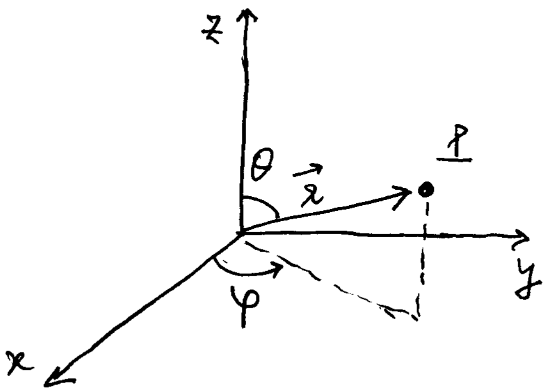
DEP. FISICA

APÊNDICE

- definição dos versores das coord. esféricas
- divergência em coordenadas esféricas

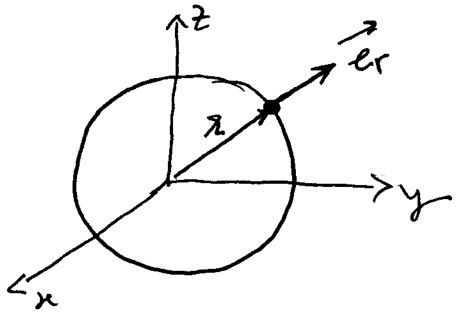
DEFINIÇÃO DOS VERSORES DAS COORD. ESFÉRICAS

As coordenadas esféricas de um ponto P envolvem uma distância (r) do ponto à origem e os ângulos polar (θ) e azimutal (φ).

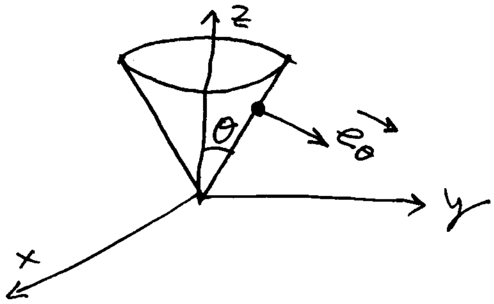


Os versores das coord. esféricas definem-se como sendo os vectores unitários perpendiculares às seguintes superfícies:

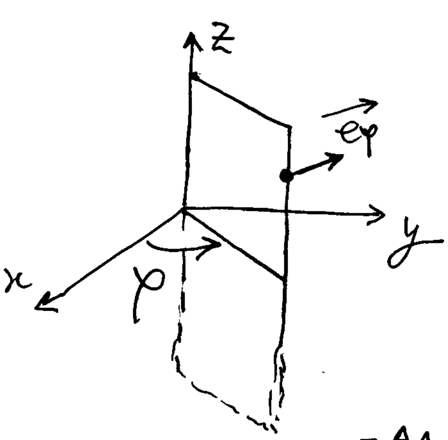
- $\vec{e_r}$: vector perpendicular à superfície $\boxed{r = c^{te}}$



- $\vec{e_\theta}$: vector perpendicular à superfície cónica $\boxed{\theta = c^{te}}$



- $\vec{e_\varphi}$: vector perpendicular à superfície (plano contendo eixo Oz) $\boxed{\varphi = c^{te}}$



DIVERGÊNCIA EM COORD. ESFÉRICAS

O campo vectorial $\vec{E}$ pode por razões de simplicidade ser expresso em coordenadas esféricas,

$$\vec{E} = E_r(r,\theta,\phi)\,\vec{e_r} + E_\theta(r,\theta,\phi)\,\vec{e_\theta} + E_\phi(r,\theta,\phi)\,\vec{e_\phi}$$

Nesse caso, deve-se tomar o operador gradiente também expresso em coordenadas esféricas,

$$\vec{\nabla} = \left(\frac{\partial}{\partial r}\right)\vec{e_r} + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\vec{e_\theta} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\vec{e_\phi}$$

e calcular então a divergência,

$$\vec{\nabla}\cdot\vec{E} = \left[\left(\frac{\partial}{\partial r}\right)\vec{e_r} + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\vec{e_\theta} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\vec{e_\phi}\right]\cdot\left[E_r\,\vec{e_r} + E_\theta\,\vec{e_\theta} + E_\phi\,\vec{e_\phi}\right]$$

O que é mais complicado que em coord. cartesianas !!!

Cálculos a realizar:

- $\vec{e_r}\,\dfrac{\partial}{\partial r}\left(E_r\,\vec{e_r}\right) = \dfrac{\partial E_r}{\partial r}$
- $\vec{e_r}\,\dfrac{\partial}{\partial r}\left(E_\theta\,\vec{e_\theta}\right) = 0$
- $\vec{e_r}\,\dfrac{\partial}{\partial r}\left(E_\phi\,\vec{e_\phi}\right) = 0$

- $\vec{e_\theta}\,\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(E_r\,\vec{e_r}\right) = \vec{e_\theta}\,\dfrac{1}{r}\left(\dfrac{\partial E_r}{\partial\theta}\,\vec{e_r} + E_r\,\dfrac{\partial\vec{e_r}}{\partial\theta}\right) = \dfrac{E_r}{r}\,\vec{e_\theta}\cdot\dfrac{\partial\vec{e_r}}{\partial\theta} = \dfrac{E_r}{r}\,\vec{e_\theta}\cdot\vec{e_\theta} = \dfrac{E_r}{r}$
- $\vec{e_\theta}\,\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(E_\theta\,\vec{e_\theta}\right) = \dfrac{1}{r}\,\vec{e_\theta}\cdot\left(\dfrac{\partial E_\theta}{\partial\theta}\,\vec{e_\theta} + E_\theta\,\underbrace{\dfrac{\partial\vec{e_\theta}}{\partial\theta}}\right) = \dfrac{1}{r}\,\dfrac{\partial E_\theta}{\partial\theta}$
- $\vec{e_\theta}\,\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(E_\phi\,\vec{e_\phi}\right) = \dfrac{1}{r}\,\vec{e_\theta}\cdot\left(\dfrac{\partial E_\phi}{\partial\theta}\,\vec{e_\phi} + E_\phi\,\underbrace{\overset{-\vec{e_r}}{\dfrac{\partial\vec{e_\phi}}{\partial\theta}}}_{0}\right) = 0$

- $\dfrac{\vec{e_\phi}}{r\sin\theta}\,\dfrac{\partial}{\partial\phi}\left(E_r\,\vec{e_r}\right) = \dfrac{1}{r\sin\theta}\,\vec{e_\phi}\cdot\left(\dfrac{\partial E_r}{\partial\phi}\,\vec{e_r} + E_r\,\underbrace{\dfrac{\partial\vec{e_r}}{\partial\phi}}_{\sin\theta\,\vec{e_\phi}}\right) = \dfrac{E_r}{r}$
- $\dfrac{\vec{e_\phi}}{r\sin\theta}\,\dfrac{\partial}{\partial\phi}\left(E_\theta\,\vec{e_\theta}\right) = \dfrac{1}{r\sin\theta}\,\vec{e_\phi}\cdot\left(\dfrac{\partial E_\theta}{\partial\phi}\,\vec{e_\theta} + E_\theta\,\underbrace{\dfrac{\partial\vec{e_\theta}}{\partial\phi}}_{\cos\theta\,\vec{e_\phi}}\right) = \dfrac{E_\theta\cos\theta}{r\sin\theta}$

$$\bullet\ \frac{\vec{e_\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\,\vec{e_\phi}\right) = \frac{1}{r\sin\theta}\,\vec{e_\phi}\cdot\left(\frac{\partial E_\phi}{\partial\phi}\vec{e_\phi} + E_\phi \underbrace{\frac{\partial\vec{e_\phi}}{\partial\phi}}_{-\cos\phi\,\vec{e_x} - \sin\phi\,\vec{e_y}}\right) =$$

$$= \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} + \frac{E_\phi}{r\sin\theta}\underbrace{\vec{e_\phi}\cdot\left(-\cos\phi\,\vec{e_x} - \sin\phi\,\vec{e_y}\right)}_{=0} =$$

$$= \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

Donde:

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_r}{\partial r} + \frac{E_r}{r} + \frac{1}{r}\frac{\partial E_\theta}{\partial\theta} + \frac{E_r}{r} + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} + \frac{E_\theta\cos\theta}{r\sin\theta}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}}$$